1. Given that

$$y = \frac{3\sin\theta}{2\sin\theta + 2\cos\theta} \qquad -\frac{\pi}{4} < \theta < \frac{3\pi}{4}$$

show that

$$\frac{dy}{d\theta} = \frac{A}{1+\sin 2\theta} \qquad -\frac{\pi}{4} < \theta < \frac{3\pi}{4}$$

where A is a rational constant to be found.

(5)

$$y = \frac{u(\theta)}{v(\theta)}, \quad \frac{dy}{d\theta} = \frac{u'(\theta)v(\theta) - u(\theta)v'(\theta)}{[v(\theta)]^2}$$

$$u(\theta) = 3\sin\theta \rightarrow u'(\theta) = 3\cos\theta$$

$$v(\theta) = 2\sin\theta + 2\cos\theta \rightarrow v'(\theta) = 2\cos\theta - 2\sin\theta$$

$$\frac{dy}{d\theta} = \frac{3\cos\theta(2\sin\theta + 2\cos\theta) - 3\sin\theta(2\cos\theta - 2\sin\theta)}{(2\sin\theta + 2\cos\theta)^2}$$ ✓✓

$$= \frac{\cancel{6\cos\theta\sin\theta} + 6\cos^2\theta - \cancel{6\cos\theta\sin\theta} + 6\sin^2\theta}{4\sin^2\theta + 4\cos^2\theta + 4\cos\theta\sin\theta + 4\cos\theta\sin\theta}$$

$$\frac{dy}{d\theta} = \frac{6(\cos^2\theta + \sin^2\theta)}{4(\cos^2\theta + \sin^2\theta) + 8\cos\theta\sin\theta}$$

$\cos^2\theta + \sin^2\theta = 1$

$2\sin\theta\cos\theta = \sin(2\theta)$

$$= \frac{6(1)}{4(1) + 4\sin(2\theta)} = \frac{6}{4 + 4\sin(2\theta)}$$ ✓✓

$$\frac{dy}{d\theta} = \frac{3}{2 + 2\sin(2\theta)} = \frac{3}{2}\left(\frac{1}{1+\sin(2\theta)}\right)$$

$$\therefore \frac{dy}{d\theta} = \frac{3/2}{1+\sin(2\theta)}$$ ✓ $\quad \therefore A = \frac{3}{2}$

2. A scientist is studying a population of mice on an island.

The number of mice, N, in the population, t months after the start of the study, is modelled by the equation

$$N = \frac{900}{3 + 7e^{-0.25t}}, \quad t \in \mathbb{R}, \quad t \geqslant 0$$

(a) Find the number of mice in the population at the start of the study.

$\hookrightarrow t = 0$

(1)

(b) Show that the rate of growth $\frac{dN}{dt}$ is given by $\frac{dN}{dt} = \frac{N(300 - N)}{1200}$

(4)

The rate of growth is a maximum after T months. $\rightarrow \frac{dN}{dt} = 0$

(c) Find, according to the model, the value of T.

(4)

According to the model, the maximum number of mice on the island is P.

(d) State the value of P. $\hookrightarrow t \rightarrow \infty$

(1)

Question continued

a) Start $\Rightarrow t = 0$

$$N_{(start)} = \frac{900}{3+7e^{-0.25(0)}}$$

$$= \frac{900}{10} = 90$$

Question continued

b)

$$N = \frac{900}{3+7e^{-0.25t}}$$

$$N = 900(3+7e^{-0.25t})^{-1}$$

$y = f(g(t)) \Rightarrow y = f(u) \quad u = g(t)$

$$\frac{dy}{dx} = f'(u) \times g'(t)$$

$N = 900u^{-1} \qquad u = 3+7e^{-0.25t}$

$\frac{dN}{du} = -900u^{-2} \qquad \frac{du}{dt} = \cancel{0} - 0.25 \times 7 \times e^{-0.25t}$

$$\frac{dN}{dt} = 900 \times 0.25 \times 7e^{-0.25t} \times (3+7e^{-0.25t})^{-2}$$

$N = \frac{900}{3+7e^{-0.25t}}, \quad 3+7e^{-0.25t} = \frac{900}{N}$

$7e^{-0.25t} = \frac{900}{N} - 3$

$$\frac{dN}{dt} = \frac{900}{4} \times \left(\frac{900}{N} - 3\right) \times \left(\frac{N^2}{900^2}\right)$$

$$= \frac{\cancel{900}}{4} \times \frac{\cancel{3}}{\cancel{N}}(300-N) \times \frac{\cancel{N^2}\ N}{\cancel{900^2}\ \cancel{900}\ 300}$$

$$= \frac{N}{300} \times \frac{1}{4} \times (300-N)$$

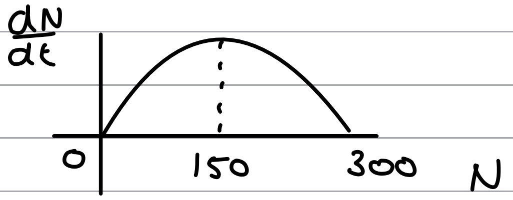
$$\therefore \frac{dN}{dt} = \frac{N(300-N)}{1200}$$

Question continued

c)

$$\frac{dN}{dt} = \frac{N(300-N)}{1200} \qquad N = \frac{900}{3+7e^{-0.25t}}$$

$$N(300-N) = 0$$



$\frac{dN}{dt}$ is maximum at $N = 150$

$$150 = \frac{900}{3+7e^{-0.25T}}$$

$$3+7e^{-0.25T} = 6$$

$$7e^{-0.25T} = 3$$

$$e^{-0.25T} = \frac{3}{7}$$

$$-0.25T = \ln\left(\frac{3}{7}\right)$$

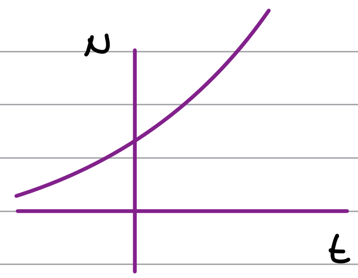
$$T = \ln\left(\frac{3}{7}\right) \times -4$$

$T = 3.38\ldots$ Months.

$T = 3.4$ Months

d)

$N = \dfrac{900}{3+7e^{-0.25t}}$



as $t \to \infty$, $0.25t \to \infty$, $-0.25t \to -\infty$

$e^{-0.25t} \to 0$

$P = \dfrac{900}{3+7(0)} = \dfrac{900}{3} = 300$ ✓

3. differentiate $\quad y = \dfrac{5x^2 + 10x}{(x+1)^2} \qquad x \neq -1$

(a) Show that $\dfrac{dy}{dx} = \dfrac{A}{(x+1)^n}$ where A and n are constants to be found.

(4)

(b) Hence deduce the range of values for x for which $\dfrac{dy}{dx} < 0$

(1)

a)

$f(x) = 5x^2 + 10x \qquad f'(x) = 10x + 10$

$g(x) = (x+1)^2 \qquad g'(x) = 2(x+1)$ 1

$2(1)(x+1)^1$

$= 2(x+1)$

$$dy/dx = \frac{(10x+10)(x+1)^{\cancel{2}\,1} - 2(5x^2+10x)(\cancel{x+1})}{(x+1)^{\cancel{4}\,3}}$$ 1

$$= \frac{(10x+10)(x+1) - 2(5x^2+10x)}{(x+1)^3}$$ 1

$$= \frac{\cancel{10x^2} + \cancel{10x} + \cancel{10x} + 10 - \cancel{10x^2} - \cancel{20x}}{(x+1)^3}$$

$$= \frac{10}{(x+1)^3}$$

$A = 10$

$n = 3$

1

a) $dy/dx = \frac{10}{(x+1)^3}$

b) $\frac{10}{(x+1)^3} < 0$

Possitive

< 0 essentially means negative

denominator must be negative

$\frac{+}{+} = + > 0$

$\frac{+}{-} = - < 0$

$(x+1)^3 < 0$

$x+1$ has to be negative

$(+)^3 = +$

$(-)^3 = -$

$x + 1 < 0$

$-1 \qquad -1$

$x < -1$ (1)

4.

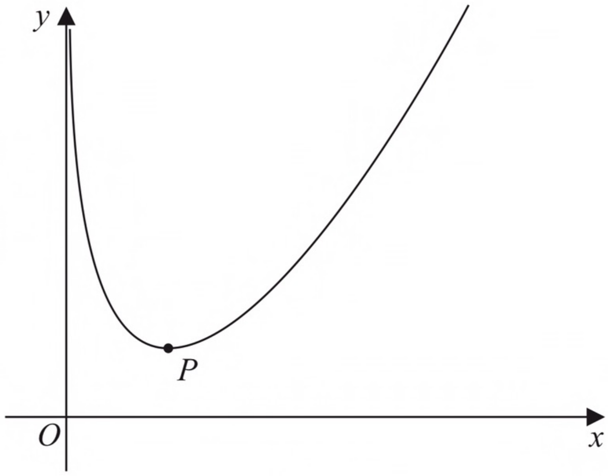


Figure 1

Figure 1 shows a sketch of the curve C with equation

$$y = \frac{4x^2 + x}{2\sqrt{x}} - 4\ln x \qquad x > 0$$

(a) Show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{12x^2 + x - 16\sqrt{x}}{4x\sqrt{x}}$$

(4)

a) $y = \underbrace{\frac{4x^2+x}{2\sqrt{x}}}_{} - \underbrace{4\ln x}_{}$, Find $\frac{dy}{dx}$

• Log Differentiation : $\frac{d}{dx}(\ln x) = \frac{1}{x}$

• Quotient Rule : If $h(x) = \frac{f(x)}{g(x)}$

then $h'(x) = \frac{f'(x)\cdot g(x) - f(x)\cdot g'(x)}{(g(x))^2}$

• $\frac{d}{dx}(4\ln x) = 4 \cdot \frac{1}{x} = \boxed{\frac{4}{x}}$ 1

let $h(x) = \frac{4x^2+x}{2\sqrt{x}} \Rightarrow f(x) = 4x^2 + x \rightarrow f'(x) = 8x + 1$

$2\sqrt{x} = 2x^{1/2}$

$g(x) = 2\sqrt{x} \rightarrow g'(x) = \frac{1}{\sqrt{x}}$ 1

$\Rightarrow h'(x) = \frac{(8x+1)(2\sqrt{x}) - (4x^2+x)(\frac{1}{\sqrt{x}})}{(2\sqrt{x})^2}$ 1 $= \frac{16x^{3/2} + 2x^{1/2} - \frac{4x^2}{x^{1/2}} - \frac{x}{x^{1/2}}}{4x} = \frac{16x^{3/2} + 2x^{1/2} - 4x^{3/2} - x^{1/2}}{4x}$

$\Rightarrow h'(x) = \frac{12x^{3/2} + x^{1/2}}{4x} = 3x^{1/2} + \frac{1}{4x^{1/2}} = \boxed{3\sqrt{x} + \frac{1}{4\sqrt{x}}}$

$\Rightarrow \frac{dy}{dx} = 3\sqrt{x} + \frac{1}{4\sqrt{x}} - \frac{4}{x} = \frac{12x+1}{4\sqrt{x}} - \frac{4}{x} = \frac{12x^2+x-16\sqrt{x}}{4x\sqrt{x}} = \frac{dy}{dx}$ as required. 1

The point P, shown in Figure 1, is the minimum turning point on C.

(b) Show that the x coordinate of P is a solution of

$$x = \left(\frac{4}{3} - \frac{\sqrt{x}}{12}\right)^{\frac{2}{3}}$$

(3)

b) From part a : $\frac{dy}{dx} = \frac{12x^2 + x - 16\sqrt{x}}{4x\sqrt{x}}$

Our first step is to set $\frac{dy}{dx} = 0$. $\Rightarrow \frac{12x^2 + x - 16\sqrt{x}}{4x\sqrt{x}} = 0$

$\Rightarrow 12x^2 + x - 16\sqrt{x} = 0$ $\div \sqrt{x}$

$\Rightarrow 12x^{3/2} + \sqrt{x} - 16 = 0$ 1

$\Rightarrow 12x^{3/2} = 16 - \sqrt{x}$ $\div 12$ 1

$\Rightarrow x^{3/2} = \frac{16}{12} - \frac{\sqrt{x}}{12}$

$\Rightarrow x^{3/2} = \frac{4}{3} - \frac{\sqrt{x}}{12}$

$\Rightarrow x = \left(\frac{4}{3} - \frac{\sqrt{x}}{12}\right)^{2/3}$ as required. 1

(c) Use the iteration formula

$$x_{n+1} = \left(\frac{4}{3} - \frac{\sqrt{x_n}}{12}\right)^{\frac{2}{3}} \qquad \text{with } x_1 = 2$$

to find (i) the value of x_2 to 5 decimal places,

(ii) the x coordinate of P to 5 decimal places.

(3)

c)

i) $x_1 = 2$ and $x_{n+1} = \left(\frac{4}{3} - \frac{\sqrt{x_n}}{12}\right)^{2/3} \Rightarrow x_2 = \left(\frac{4}{3} - \frac{\sqrt{x_1}}{12}\right)^{2/3} = \left(\frac{4}{3} - \frac{\sqrt{2}}{12}\right)^{2/3}$ 1

Sub this in!

$x_2 = 1.138935...$

$x_2 = 1.13894$ (5 d.p) 1

ii) $x = 1.15650$ 1

5. The function g is defined by

$$g(x) = \frac{3\ln(x) - 7}{\ln(x) - 2} \qquad x > 0 \qquad x \neq k$$

where k is a constant.

(a) Deduce the value of k.

(1)

• When we have a fraction, the denominator cannot equal 0,

$\Rightarrow \quad \ln x - 2 = 0$

$\ln x = 2$

$e^{\ln x} = e^2$

$x = e^2 \qquad \Rightarrow \quad k = e^2$ or $x \neq e^2$ ①

(b) Prove that

$$g'(x) > 0$$

for all values of x in the domain of g.

(3)

Quotient Rule If $g(x) = \frac{f(x)}{h(x)}$ then $g'(x) = \frac{f'(x)h(x) - f(x)h'(x)}{(h(x))^2}$

Recall that $g(x) = \frac{3\ln x - 7}{\ln x - 2} \Rightarrow$ let $f(x) = 3\ln x - 7$ then $f'(x) = \frac{3}{x}$ ①

$h(x) = \ln x - 2$ then $h'(x) = \frac{1}{x}$

$$\Rightarrow g'(x) = \frac{\frac{3}{x}(\ln x - 2) - \frac{1}{x}(3\ln x - 7)}{(\ln x - 2)^2} = \frac{\frac{3}{x}\cdot\ln x - \frac{6}{x} - \frac{3\ln x}{x} + \frac{7}{x}}{(\ln x - 2)^2} = \frac{1/x}{(\ln x - 2)^2}$$

$$\Rightarrow g'(x) = \frac{1}{x(\ln x - 2)^2} \quad ①$$

• We know that $x > 0$

• $(\ln x - 2)$ is squared

$\Rightarrow$ the denominator is always positive, hence $g'(x) > 0$. ①

(c) Find the range of values of a for which

$$g(a) > 0$$

(2)

Recall that $g(x) = \dfrac{3\ln x - 7}{\ln x - 2}$

let $\ln x = y$, then $g(x) = \dfrac{3y-7}{y-2} > 0$.

• Multiply both sides by $(y-2)$ $\Rightarrow$ $3y - 7 > 0$. 1

$\Rightarrow$ $y > 7/3$

$\Rightarrow$ $\ln x > 7/3$ (we change x to a now)

$\Rightarrow$ $\underline{a > e^{7/3}}$

• $y = 2$ then $g(x)$ not defined since denominator equal to 0.

$\Rightarrow y < 2$

$\Rightarrow \ln(a) < 2$

$\Rightarrow a < e^2$ $\Rightarrow$ $\underline{0 < a < e^2}$ and $\underline{a > e^{7/3}}$. 1

6. Given that

$$y = \frac{x-4}{2+\sqrt{x}} \qquad x > 0$$

show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{A\sqrt{x}} \qquad x > 0$$

where A is a constant to be found.

(4)

$$y = \frac{x-4}{2+x^{1/2}}$$

$u = x - 4 \qquad v = 2 + x^{1/2}$

$u' = 1 \qquad v' = \frac{1}{2}x^{-1/2}$

$$\frac{dy}{dx} = \frac{(2+x^{1/2})(1) - (x-4)(\frac{1}{2}x^{-1/2})}{(2+x^{1/2})^2} \quad 1 \quad 1$$

$$= \frac{2+\sqrt{x} - \frac{1}{2}x^{1/2} + 2x^{-1/2}}{(2+x^{1/2})^2}$$

$$= \frac{2+\sqrt{x} - \frac{\sqrt{x}}{2} + \frac{2}{\sqrt{x}}}{(2+\sqrt{x})^2}$$ → multiply the numerator by $\sqrt{x}$

$$= \frac{2\sqrt{x} + x - \frac{x}{2} + 2}{\sqrt{x}(2+\sqrt{x})^2}$$

$$= \frac{2\sqrt{x} + \frac{x}{2} + 2}{\sqrt{x}(2+\sqrt{x})^2}$$ 1 → multiply numerator by 2

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Question continued

$$= \frac{4\sqrt{x} + x + 4}{2\sqrt{x}(2+\sqrt{x})^2}$$

$$= \frac{\cancel{4\sqrt{x} + x + 4}}{2\sqrt{x}\,(\cancel{4 + 4\sqrt{x} + x})}$$

$$= \frac{1}{2\sqrt{x}}$$ 1

$\therefore A = 2$ #

7.

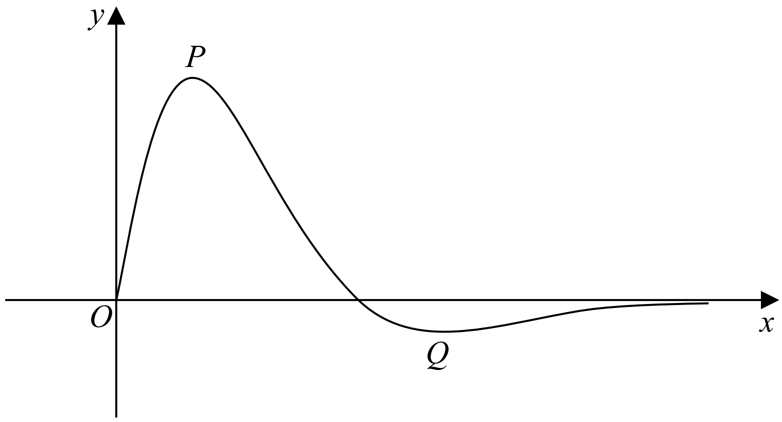

Figure 5

Figure 5 shows a sketch of the curve with equation $y = \text{f}(x)$, where

$$\text{f}(x) = \frac{4\sin 2x}{\text{e}^{\sqrt{2}x-1}}, \quad 0 \leqslant x \leqslant \pi$$

The curve has a maximum turning point at P and a minimum turning point at Q as shown in Figure 5.

(a) Show that the x coordinates of point P and point Q are solutions of the equation

$$\tan 2x = \sqrt{2}$$

(4)

(b) Using your answer to part (a), find the x-coordinate of the minimum turning point on the curve with equation

(i) $y = \text{f}(2x)$.

(ii) $y = 3 - 2\text{f}(x)$.

(4)

Question continued

a) $f(x) = \dfrac{4\sin 2x}{e^{\sqrt{2}x-1}}$

Quotient Rule : 1

$f'(x) = \dfrac{h'(x)\cdot g(x) - h(x)\cdot g'(x)}{(g(x))^2}$

• Stationary Point when $\boxed{f'(x) = 0}$

let $h(x) = 4\sin 2x$ $\quad h'(x) = 8\cos 2x$

$g(x) = e^{\sqrt{2}x-1}$ $\quad g'(x) = \sqrt{2}e^{\sqrt{2}x-1}$

$\Rightarrow f'(x) = \dfrac{8\cos 2x \cdot e^{\sqrt{2}x-1} - 4\sin 2x \cdot \sqrt{2}e^{\sqrt{2}x-1}}{(e^{\sqrt{2}x-1})^2} = 0$ 1

$\Rightarrow 8\cos 2x \cdot e^{\sqrt{2}x-1} - 4\sqrt{2}\sin 2x e^{\sqrt{2}x-1} = 0$

$\Rightarrow e^{\sqrt{2}x-1}(8\cos 2x - 4\sqrt{2}\sin 2x) = 0$ 1

$\Rightarrow 8\cos 2x - 4\sqrt{2}\sin 2x = 0$

$\Rightarrow 8\cos 2x = 4\sqrt{2}\sin 2x$

$\Rightarrow 8 = \dfrac{4\sqrt{2}\sin 2x}{\cos 2x} \Rightarrow \dfrac{\sin 2x}{\cos 2x} = \dfrac{8}{4\sqrt{2}}$

* $\tan 2x = \dfrac{\sin 2x}{\cos 2x}$

$\Rightarrow \tan(2x) = \sqrt{2}$ as required 1

b) i) $y = f(2x)$

For $y = f(x) \Rightarrow \tan 2x = \sqrt{2}$

For $y = f(2x) \Rightarrow \tan 4x = \sqrt{2}$

$x = \dfrac{\tan^{-1}\sqrt{2}}{4} + \dfrac{\pi}{4}$ 1

$x = 1.024$ 1

ii) $y = 3 - 2f(x) \Rightarrow \tan 2x = \sqrt{2}$

$x = \dfrac{\tan^{-1}(\sqrt{2})}{2} = 0.478$ 1 1

8. Given $y = x(2x+1)^4$, show that

$$\frac{dy}{dx} = (2x+1)^n(Ax+B)$$

where n, A and B are constants to be found.

(4)

If $y = uv$, $\frac{dy}{dx} = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$

$u = x$ $\quad$ $v = (2x+1)^4$

$\frac{du}{dx} = 1$ $\quad$ $\frac{dv}{dx} = 4(2x+1)^3 \times 2$ - 1

$= 8(2x+1)^3$

$x \times 8(2x+1)^3 + (2x+1)^4 \times 1$

$= 8x(2x+1)^3 + (2x+1)^4$ - 1

$= (2x+1)^3(8x + 2x+1)$ - 1

$= (2x+1)^3(10x+1)$

$n = 3$

$A = 10$ - 1

$B = 1$

9.

$$\frac{1+11x-6x^2}{(x-3)(1-2x)} \equiv A + \frac{B}{(x-3)} + \frac{C}{(1-2x)}$$

(a) Find the values of the constants A, B and C.

(4)

$$f(x) = \frac{1+11x-6x^2}{(x-3)(1-2x)} \qquad x > 3$$

(b) Prove that f(x) is a decreasing function. $\rightarrow f'(x) < 0$

(3)

Question continued

a)

$$\frac{a}{b} + \frac{c}{d} + \frac{e}{f} = \frac{adf + cbf + ebf}{bdf}$$

$$\frac{1 + 11x - 6x^2}{(x-3)(1-2x)} = \frac{A(x-3)(1-2x) + B(1-2x) + C(x-3)}{(x-3)(1-2x)}$$

$$\frac{a}{b} = \frac{c}{b} \Rightarrow a = c$$

$$1 + 11x - 6x^2 = A(x-3)(1-2x) + B(1-2x) + C(x-3)$$

To find B, $x - 3 = 0 \Rightarrow x = 3$

$$1 + 11(3) - 6(3)^2 = 0 + B(-5)$$

$$-20 = -5B \Rightarrow B = 4$$

To find C, $1 - 2x = 0 \Rightarrow 2x = 1 \quad \therefore x = 0.5$

$$1 + 11(0.5) - 6(0.5)^2 = 0 + 0 - 2.5C$$

$$5 = -2.5C$$

$$C = -2$$

$$1 + 11x - 6x^2 = A(x-3)(1-2x) + 4(1-2x) - 2(x-3)$$

$x = 0$, $\quad 1 + 11(0) - 6(0) = -3A + 4 + 6$

$$1 = -3A + 10$$

$$-3A = -9$$

$$A = 3$$

$$\frac{1 + 11x - 6x^2}{(x-3)(1-2x)} \equiv 3 + \frac{4}{(x-3)} - \frac{2}{(1-2x)}$$

Question continued

b)

$f(x) = 3 + \frac{4}{(x-3)} - \frac{2}{(1-2x)}$

if $y = A(f(x))^n$
$\frac{dy}{dx} = Anx(f'(x))^{n-1}$

$f(x) = 3 + 4(x-3)^{-1} - 2(1-2x)^{-1}$

$a^{-n} = \frac{1}{a^n}$

$f'(x) = 0 - 4(x-3)^{-2} - 4(1-2x)^{-2}$ ✓✓

$= \frac{-4}{(x-3)^2} + \frac{-4}{(1-2x)^2}$

$(x-3)^2 > 0$ for all $x > 3$
$(1-2x)^2 > 0$ for all $x > 3$

$\therefore -\frac{4}{(x-3)^2} < 0$ and $-\frac{4}{(1-2x)^2} < 0$

We have shown that $-\frac{4}{(x-3)^2}$ and $-\frac{4}{(1-2x)^2} < 0$ for all $x > 3$. $f'(x)$ is the sum of 2 negative fractions. $\therefore f'(x) < 0 \Rightarrow f(x)$ is decreasing. ✓